Hybrid Gibbs Sampling and MCMC for CMB Analysis at Small Angular Scales

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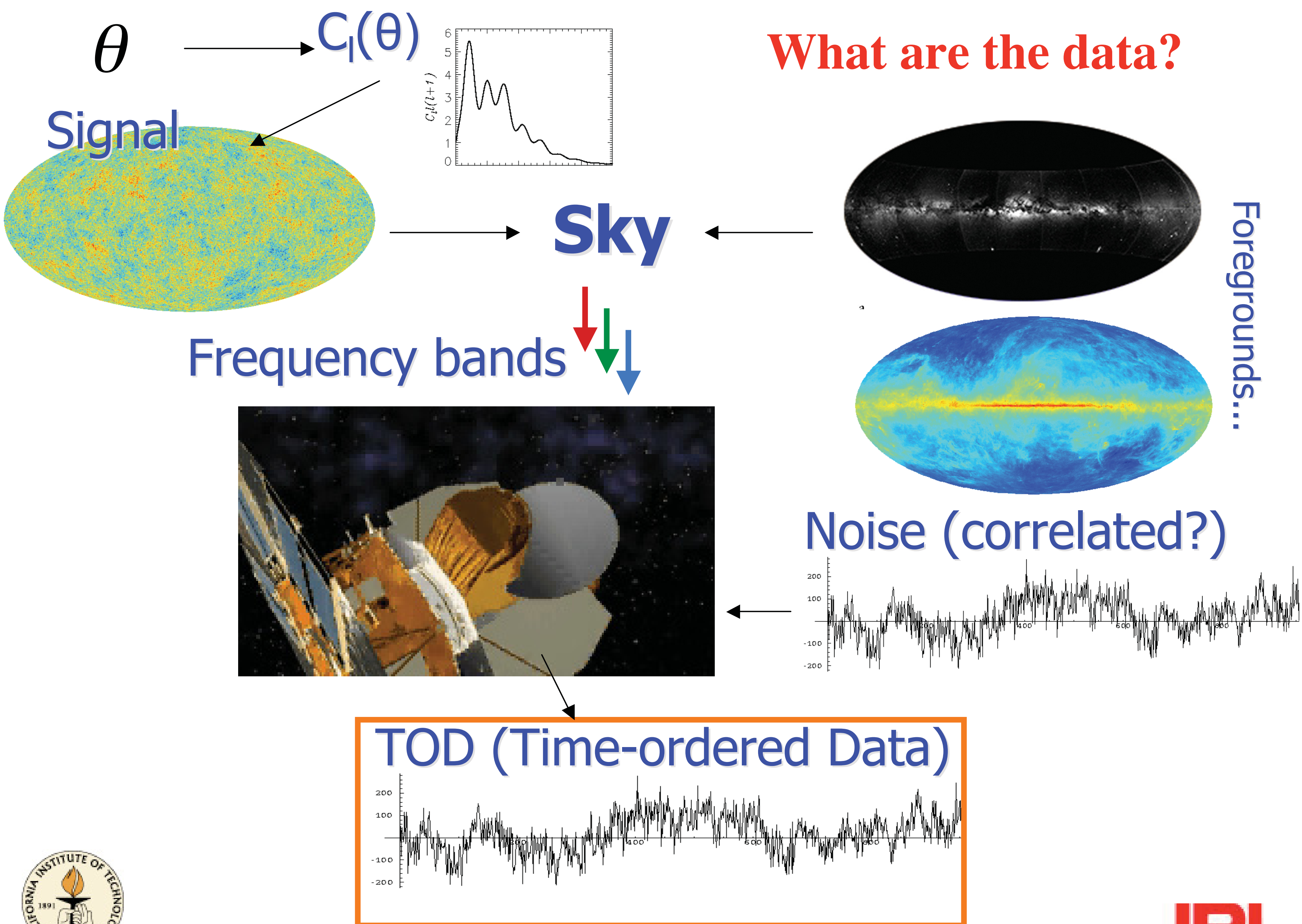
What are the data?



Simulation and Inference

Joint density of "everything":

$$p(d,s,\theta) = p(d\,|\,s)\,p(s\,|\,\theta)\,p(\theta)$$

Data Underlying "truth" Model parameters

Simulation: Condition on the model

$$p(d,s\,|\,\theta) = p(d\,|\,s)\,p(s\,|\,\theta)$$

Inference: Condition on the data

$$p(\theta,s\,|\,d) \propto p(d\,|\,s)\,p(s\,|\,\theta)\,p(\theta)$$

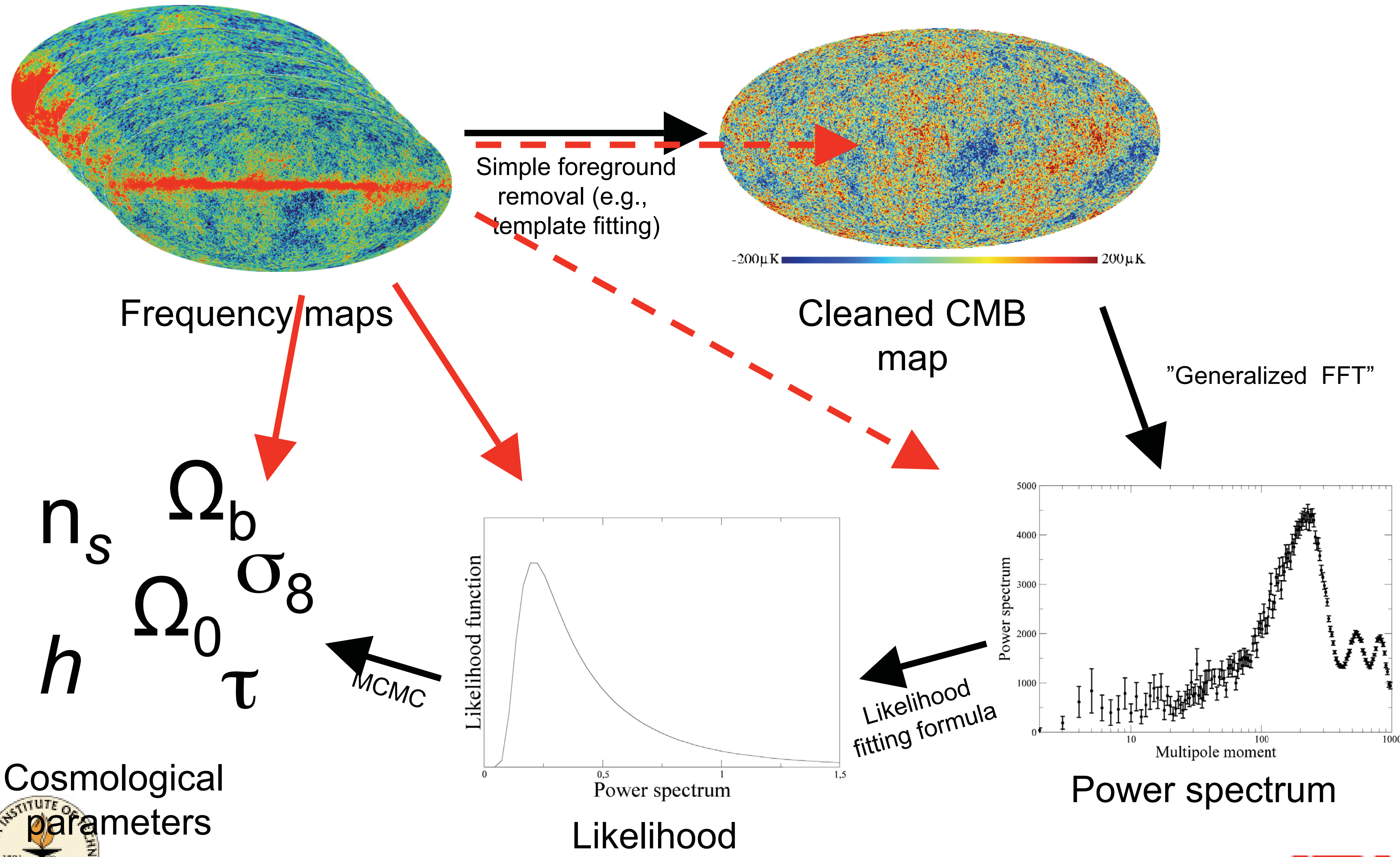
Factors in joint density given CMB data:

$$-2\log p(d\,|\,s) \propto (d_\nu - A_\nu s)N_\nu^{-1}(d_\nu - A_\nu s) = \chi^2$$

$$-2\log p(s\,|\,\theta) = sC^{-1}(\theta)s + \log\left|C(\theta)\right|$$

$$p(\theta) = \text{Prior on parameters}$$

Bayesian CMB Analysis - Can we Beat O(N^3)??



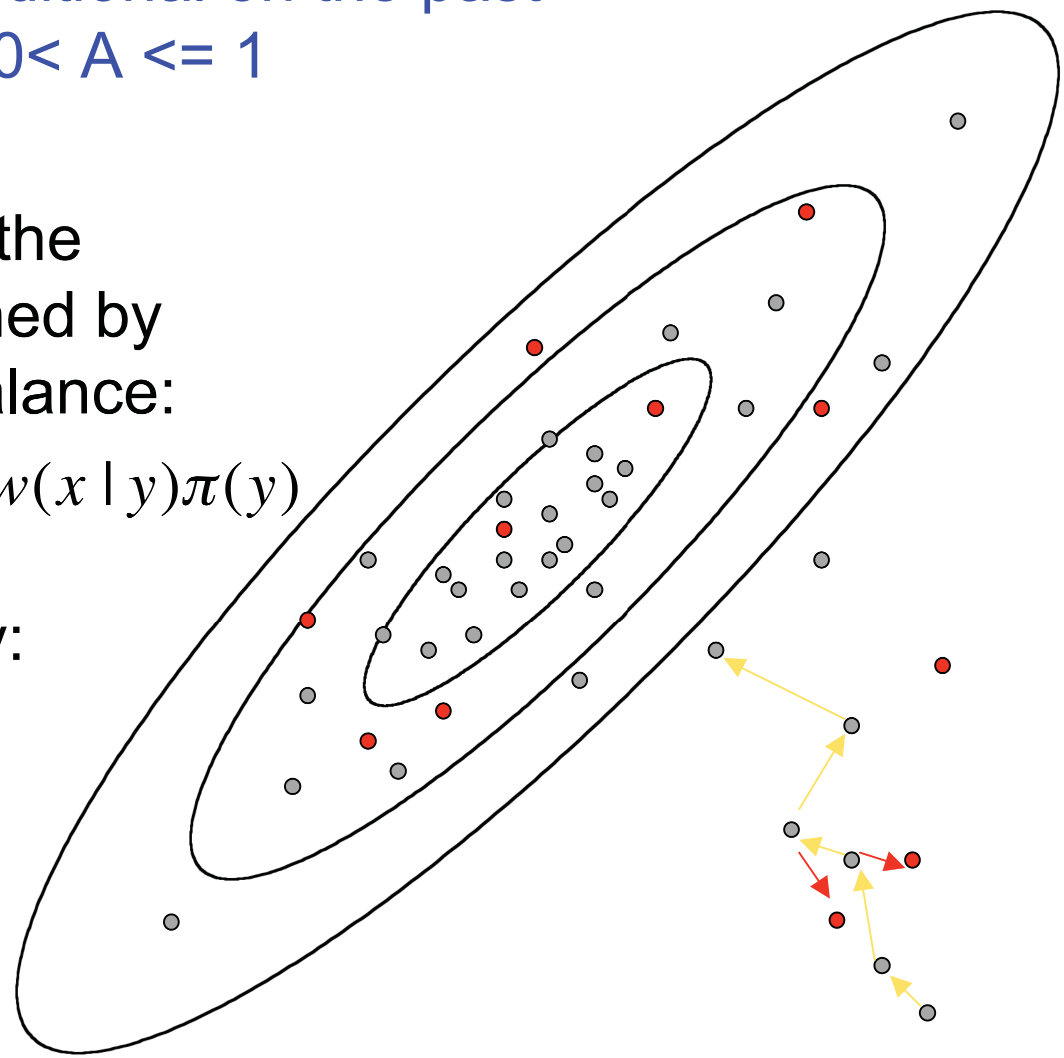
Mapping the Posterior with Metropolis-Hastings MCMC

Algorithm:

1) Propose new state, conditional on the past
2) Accept with probability 0< A <= 1
3) Continue

For any "proposal" matrix, the accept probability determined by the condition of detailed balance:

$$\pi(x)w(y \mid x)A(y \mid x) = A(x \mid y)w(x \mid y)\pi(y)$$

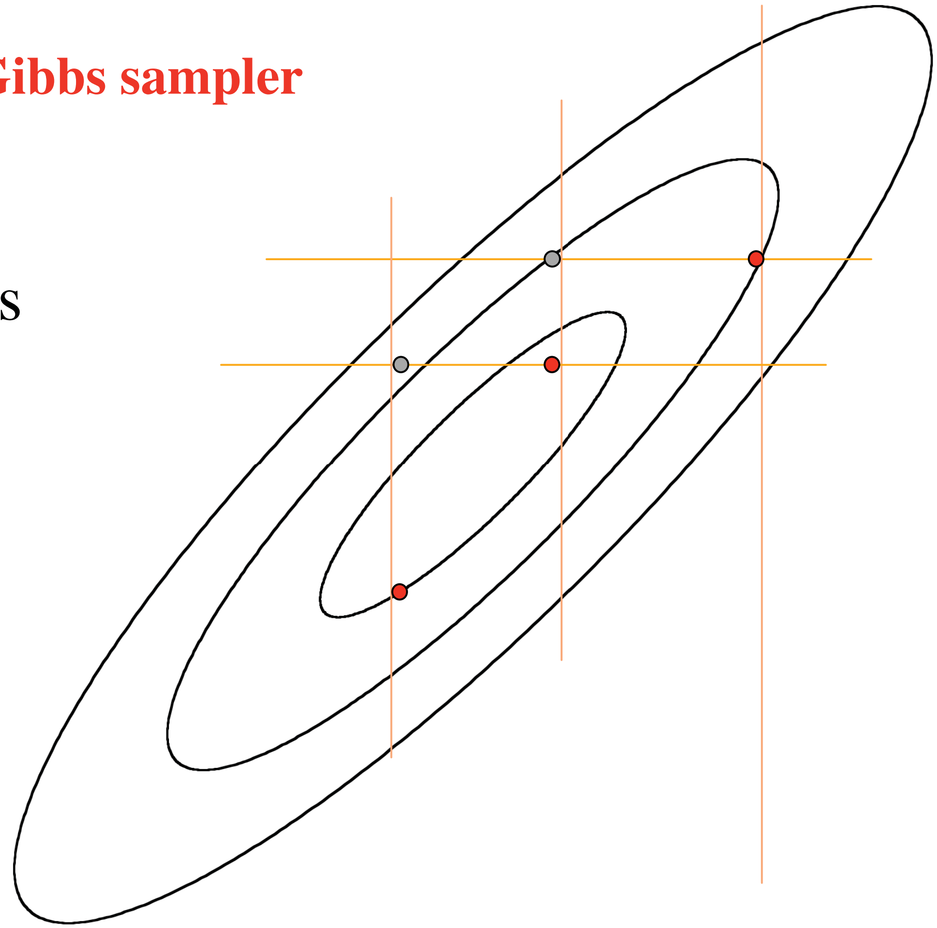


Maximal Accept Probability:

$$A(y \mid x) = \min\left[1, \frac{\pi(y)w(x \mid y)}{\pi(x)w(y \mid x)}\right]$$

Special Case of MH MCMC: The Gibbs sampler

-Sequentially propose variations from conditional densities…
-Accept probability is unity!!



Gibbs Sampling for CMB:

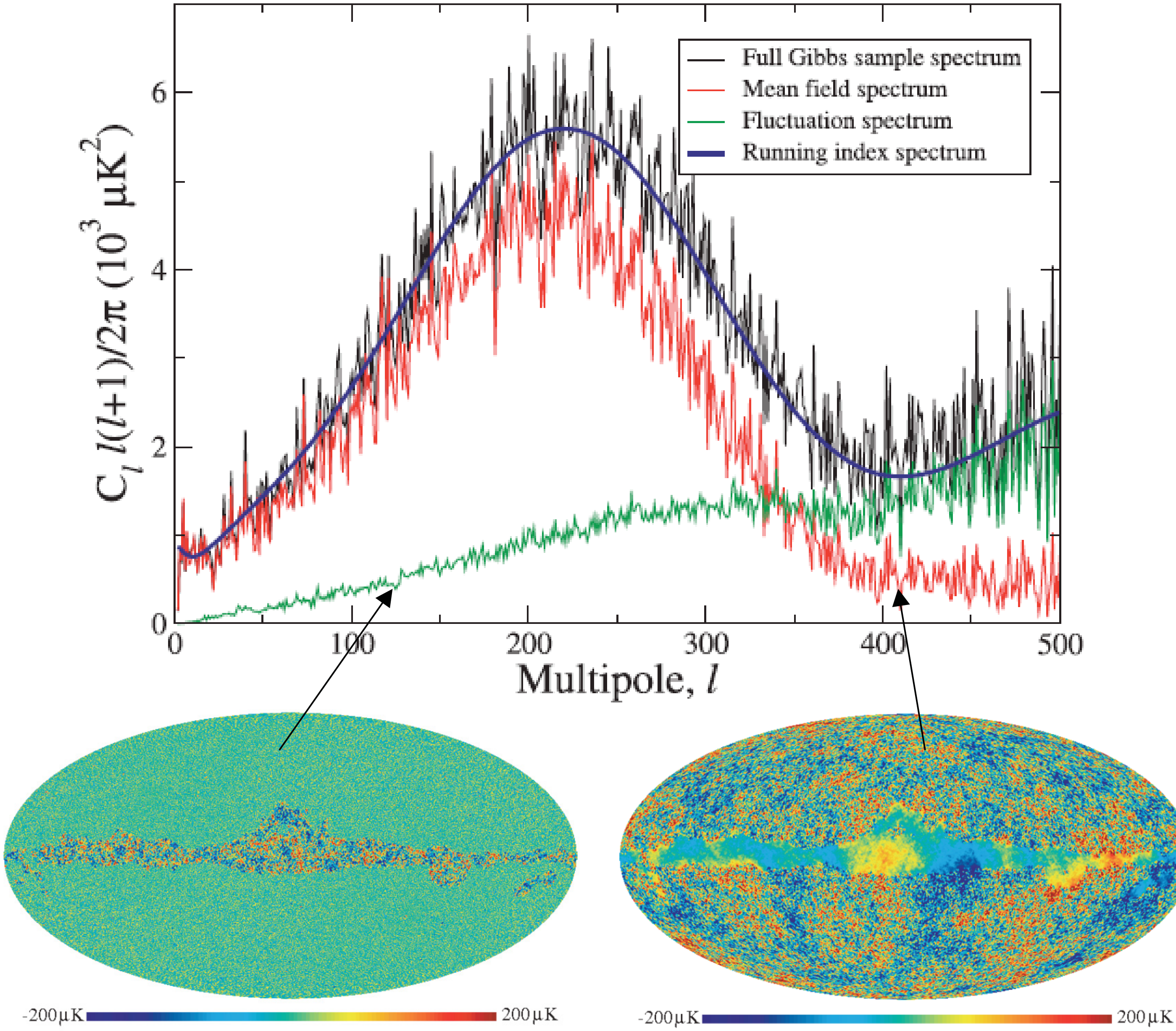
$$s^{(i+1)} \leftarrow p(s \mid C_l^{(i)}, d)$$

$$C_l^{(i+1)} \leftarrow p(C_l \mid s^{(i+1)}, d) = p(C_l \mid s^{(i+1)})$$

Method originally presented in:

- Jewell, et al., ApJ, 609,1,2004
- Wandelt et al., Phys. Rev. D., 70,083511,2004

Sampling the CMB given the Power Spectrum and Data

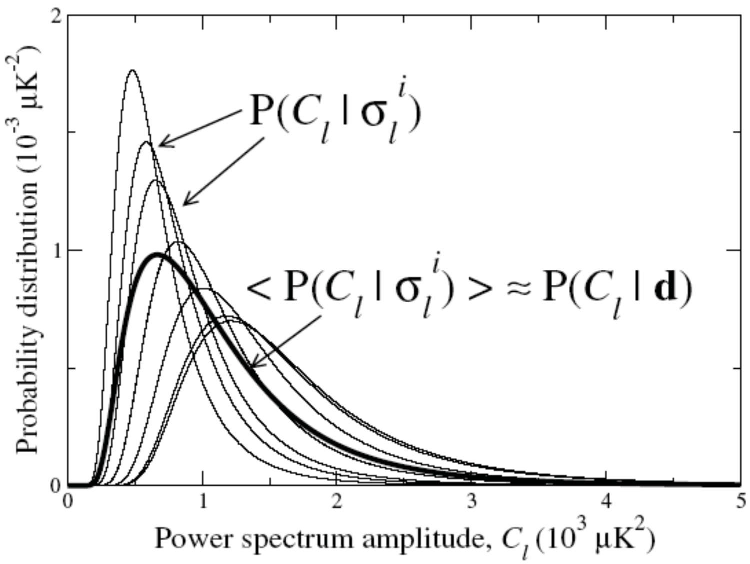


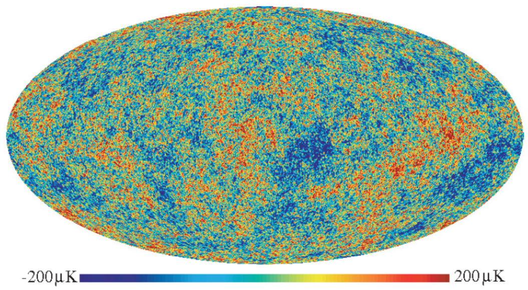
J. Jewell, 2008 Aspen Winter Conference

CMB Gibbs Sampler

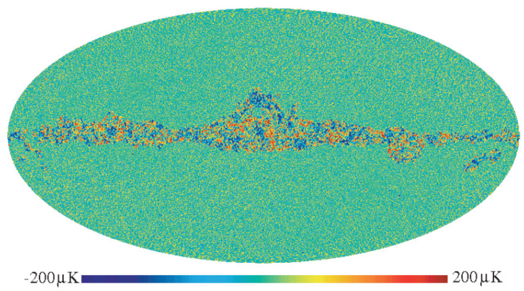
Iterate with: $$p(\theta \mid s)\, p(s \mid d, \theta')$$

$$p(\theta \mid s) \propto p(\theta) \prod_{lm} \frac{e^{-\sigma_l / 2C_l(\theta)}}{\sqrt{2\pi} C_l^{1/2}(\theta)}$$

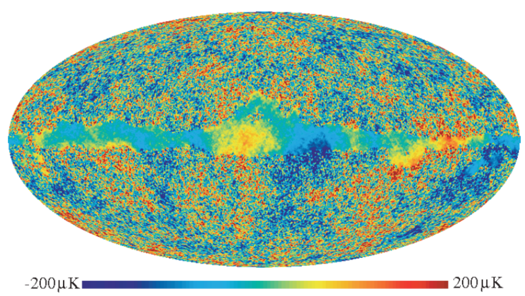




Sum of the two maps is a sample from the conditional



Random variation consistent with our uncertainty



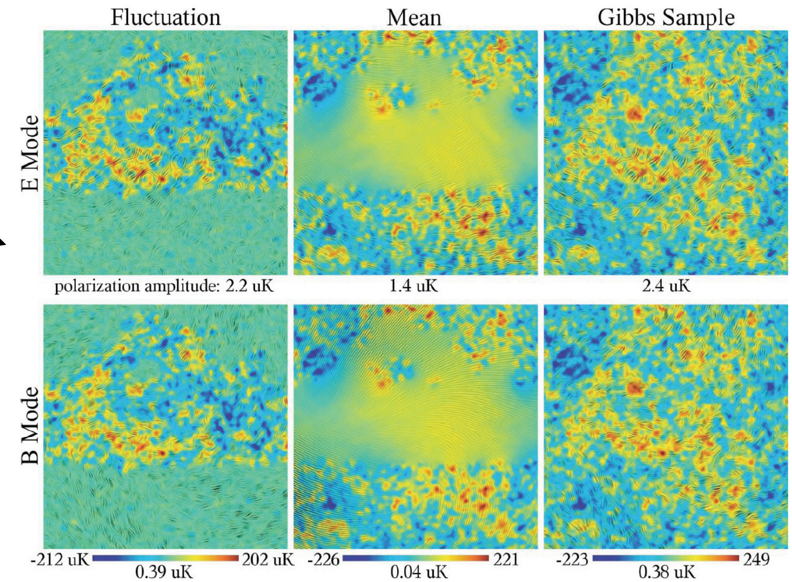
Mean Field map given power spectrum guess

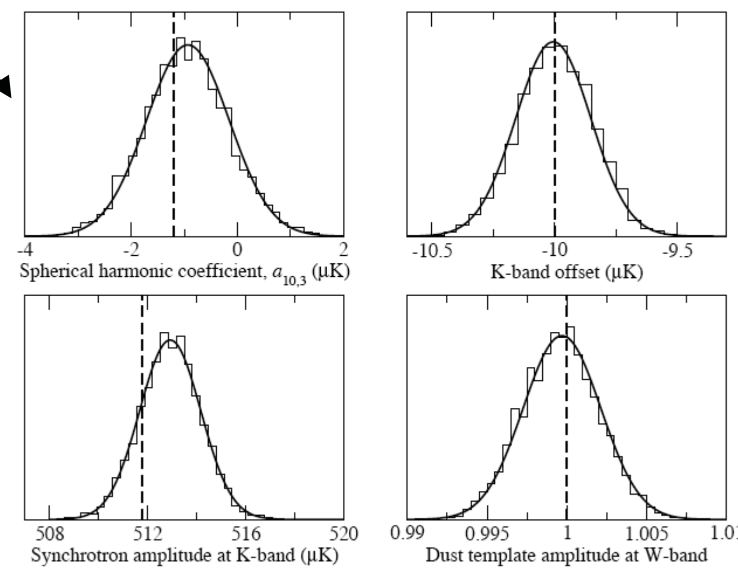
Validation and Applications

- Validation for temperature and polarization:
  - *Power Spectrum Estimation from High-Resolution Maps by Gibbs Sampling*, Eriksen et al., ApJS, 155, 227, 2004
  - *Estimation of Polarized Power Spectra by Gibbs Sampling*, Larson et al., ApJ, 656, 653, 2007
- Extension of method to include foregrounds (temperature data):
  - *Joint Bayesian Component Separation and CMB Power Spectrum Estimation*, Eriksen et al., accepted to ApJ, arXiv 0709.1058
- Applications to WMAP data:
  - *Bayesian Power Spectrum Analysis of the First-Year Wilkinson Microwave Anisotropy Probe Data*, O'Dwyer et al., ApJL, 617, 99, 2004
  - *A Reanalysis of the 3 Year Wilkinson Anisotropy Probe Temperature Power Spectrum and Likelihood*, Eriksen et al., ApJ, 656, 641, 2007
  - *Bayesian Analysis of the Low-Resolution Polarized 3 Year WMAP Sky Maps*, Eriksen et al., ApJL, 665, 1, 2007

- Joint CMB and Foreground analysis of WMAP 3 yr. data:
  - Temperature only - **see Clive Dickinson's talk**, as well as Eriksen et al., ApJL, in press, arXiv 0709.1037
  - Temp. and Polarizaton - **see H.K.K. Eriksen's talk**

- From Gibbs samples to cosmological parameters
  - Chu et al., 2005, Phys. Rev. D., 71, 103002





Comparison of Computational Expense

Direct evaluation: Computational Expense: $O[N^3]$

$$-\log \frac{p(\theta \mid d)}{p(\theta)} = \hat{s}(d)[C(\theta)+N]^{-1}\hat{s}(d) + \log\left\|C(\theta)+N\right\|$$

Gibbs Sampling: Computational Expense: $KO[N^{3/2}]$

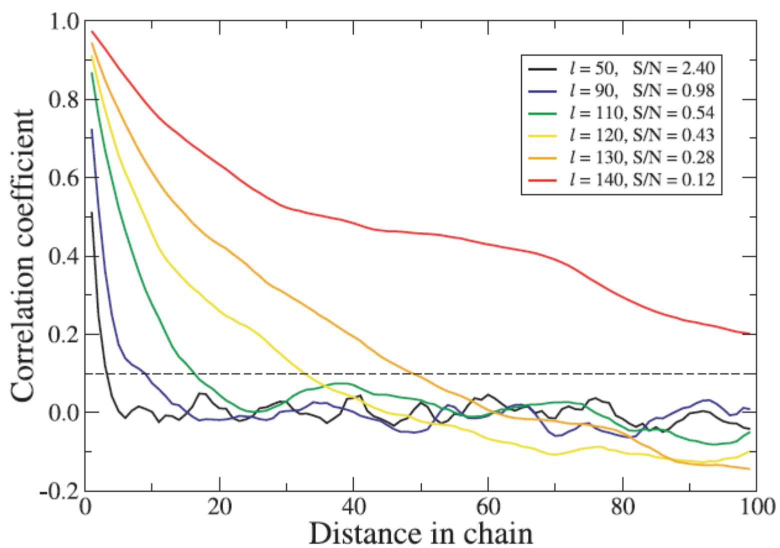
$$\boxed{p(\theta \mid d) \leftarrow_{\infty} \int d\theta' \left[\int ds\; p(\theta \mid s,d)\, p(s \mid \theta',d)\right] p_0(\theta' \mid d)}$$

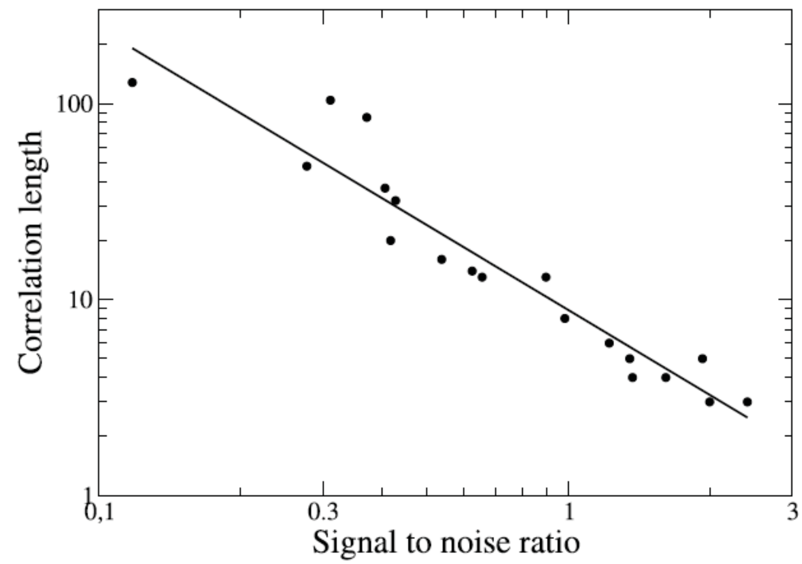
Map-making is the computational bottleneck (scales with expense of mutliplication by N^-1)

Some specific benchmark numbers:

1) Including foregrounds (T only, 5 frequencies = # processors), Nside=64, takes 50 sec/ sample/ freq., or 5000 samples in 350 CPU hours
2) Polarization, Nside=16, dense noise matrix, 2 sec/ sample, or 10^5 samples in 60 CPU hours

Low Signal to Noise, High-L Mixing Properties of Gibbs Sampling





- We want independent samples from joint posterior at all scales
- Gibbs sampling p(C_l | s) at high-L (low S/N) very narrowly peaked (cosmic variance - instead of cosmic AND noise variance)
- Attempting to propose large C_L changes in MCMC, independent of past typically lead to ratio's of matrix determinants which are too expensive to compute…
- Motivates a search for a scheme in which large changes in spectrum can be made with deterministic changes to the CMB map!

Example - Rescale Harmonic Coefficients

"Forward" proposal for CMB map: $s^{(2)} = F(C_l^{(2)}, C_l^{(1)}, s^{(1)}) = (C^{(2)})^{1/2}(C^{(1)})^{-1/2} s^{(1)}$

"Backward" proposal for CMB map: $s^{(1)} = F^{-1}(C_l^{(2)}, C_l^{(1)}, s^{(2)}) = (C^{(1)})^{1/2}(C^{(2)})^{-1/2} s^{(2)}$

Jacobian Factor to be included in Accept Probability:

$$\left|\frac{\partial F}{\partial s^{(2)}}\right|^{-1} = \left|\frac{C^{(2)}}{C^{(1)}}\right|^{1/2}$$

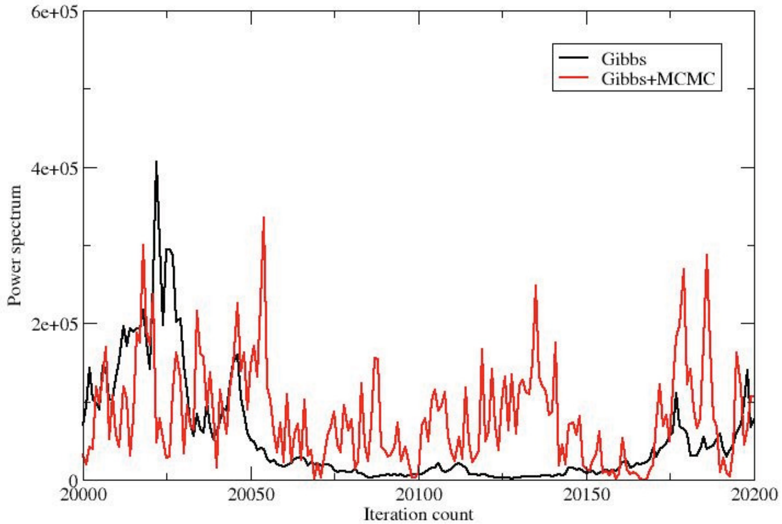
Cancels ratio of determinants in posterior!!

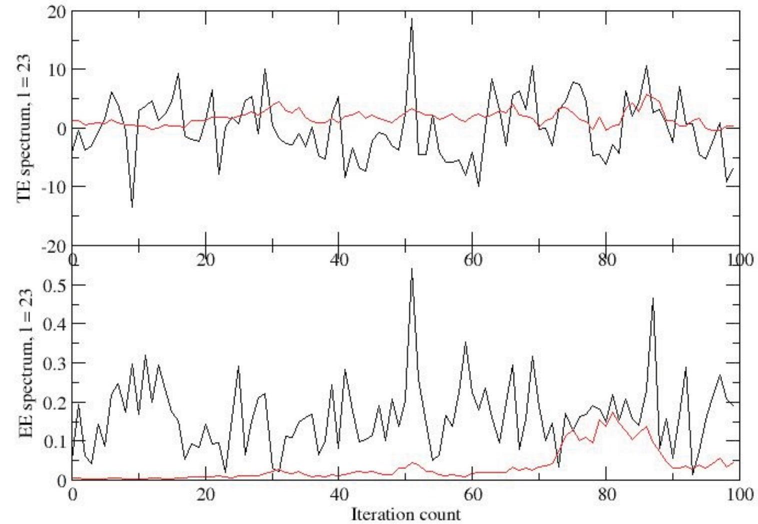
Furthermore - signal "norm" invariant: $s^{(1)}(C^{(1)})^{-1} s^{(1)} = s^{(2)}(C^{(2)})^{-1} s^{(2)}$

$$\boxed{A(C_l^{(2)}, s^{(2)} \mid C_l^{(1)}, s^{(1)}) = \min\left[1, \left(\frac{e^{-(d-s^{(2)})N^{-1}(d-s^{(2)})}}{e^{-(d-s^{(1)})N^{-1}(d-s^{(1)})}}\right) \frac{w(C_l^{(1)} \mid s^{(2)}, C_l^{(2)}, d)}{w(C_l^{(2)} \mid s^{(1)}, C_l^{(1)}, d)}\right]}$$

1. A(1-> 2) depends on change in chi^2!
2. So make large changes to C_L in low S/N regime:
   where standard Gibbs sampling has bad mixing properties!!

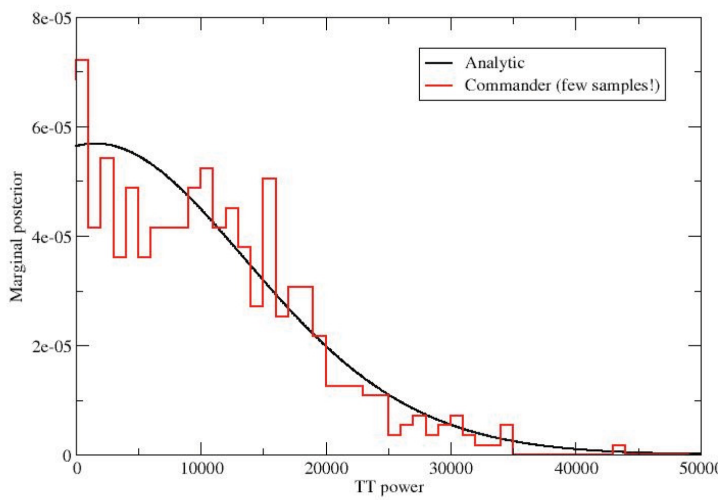
Comparison of Hybrid MCMC+Gibbs to Standard Gibbs





Hybrid MCMC + Gibbs Sampling: Left) Comparison of Gibbs and Gibbs+MCMC power vs. iteration, Right) Comparison for TE and EE power at L=23

New Hybrid MCMC and Gibbs Sampling (L=220 Marginal shown…)



Summary

- Gibbs Sampling has now been validated as an efficient, statistically exact, and practically useful method for "low-L" (as demonstrated on WMAP temperature polarization data)
- *We are extending Gibbs sampling to directly propagate uncertainties in both foreground and instrument models to total uncertainty in cosmological parameters for the entire range of angular scales relevant for Planck*
- Made possible by inclusion of foreground model parameters in Gibbs sampling and hybrid MCMC and Gibbs sampling for the low signal to noise (high-L) regime
- Future items to be included in the Bayesian framework include:
  1. Integration with Hybrid Likelihood (or posterior) code for cosmological parameters
  2. Include other uncertainties in instrumental systematics? (I.e. beam uncertainties, noise estimation, calibration errors, other)